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The Dirac operator on Lorentzian spin manifolds and the Huygens property

Helga Baum [1]

Humboldt-Universitat Zu Berlin, Institut fur Reine Mathematik, Sitz: Zeigelstrasse 13a, 10099 Berlin, Germany

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Abstract

We consider the Dirac operator D of a Lorentzian spin manifold of even dimension $n \geq 4$. We prove that the square D^2 of the Dirac operator on plane wave manifolds and the shifted operator $D^2 - K$ on Lorentzian space forms of constant sectional curvature K are of Huygens type. Furthermore, we study the Huygens property for coupled Dirac operators on four-dimensional Lorentzian spin manifolds.

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1. Introduction

It is a familiar phenomenon that waves propagate quite different in two and three dimensions. When a pebble falls into water at a certain point x_0, circular waves around x_0 are formed. A given point near x_0 will be hit by an initial ripple and later by residual waves. Three dimensionally, the situation is quite different. If we produce a sound localized at the neighbourhood of a point x_0 then someone near x_0 will hear the sound during a certain time interval but no longer. There are no residual waves like those present on the water surface.

The mathematical reason for this different behaviour is a special property of the fundamental solution of the wave operator $\Box_m$ of the $\mathbb{R}^m$ in dimension $m = 3$. Whereas in general the forward fundamental solution of $\Box_m$ with respect to the point $o \in \mathbb{R}^{m+1}$ is supported in

[1] E-mail: baum@mathematik.hu_berlin.de.

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the future cone $\mathcal{J}_+(o) = \{(x, t) \in \mathbb{R}^m \times \mathbb{R} \mid \|x\| \leq t\}$ the forward fundamental solution in dimension $m = 3$ and each other odd dimension $m \geq 3$ is supported even in the light cone $C_+(o) = \{(x, t) \in \mathbb{R}^m \times \mathbb{R} \mid \|x\| = t\}$. This produces a "sharp" wave propagation. Operators describing a "sharp" wave propagation such as $\Box_3$ are called operators of Huygens type or shortly Huygens operators. In 1923, in his Yale Lectures, Hadamard posed the problem of finding all normally hyperbolic operators of Huygens type (see [Had23, p.236]). In spite of its age this problem is still far from being completely solved. For results of the last 30 years and methods developed to treat this problem see [Gün88,Gün91,Wün94,BV94,BK96].

In this paper we consider the Huygens property for the square of the Dirac operator of a Lorentzian spin manifold.

Dirac operators on four-dimensional analytic Lorentzian spin manifolds were studied by Wünsch. He proved the following result.

Theorem 1 ([Wün 78, Corrollary 3.3; Wün 79, Proposition 5.6; Wün80, Proposition 2.1]). *Let us denote by $D : \Gamma(S) \longrightarrow \Gamma(S)$ the Dirac operator of a four-dimensional analytic Lorentzian spin manifold (M^4, g) and let f be a smooth function on M. If the operator $D^2 - f : \Gamma(S) \longrightarrow \Gamma(S)$ is of Huygens type, then the scalar curvature R of (M, g) is constant and equals $12f$.*

If R is constant and non-zero, then $D^2 - \frac{1}{12}R$ is of Huygens type if and only if (M^4, g) has constant sectional curvature.

If R is identically zero, then D^2 is of Huygens type if and only if (M^4, g) is conformally flat or a plane wave manifold.

According to Theorem 1 there are exactly three classes of four-dimensional analytic Lorentzian spin manifolds on which D^2 (or a shift of it) is of Huygens type: the Lorentzian space forms, the plane wave manifolds and the conformally flat manifolds of vanishing scalar curvature.

In this paper we consider Lorentzian spin manifolds (M^n, g) of even dimension $n \geq 4$. Let us denote by D the Dirac operator of (M^n, g). We prove that the operator $D^2 - K$ is of Huygens type if (M^n, g) has constant sectional curvature K (Theorem 7) and that D^2 is of Huygens type if (M^n, g) is a plane wave manifold (Theorem 10). It would be interesting to know whether D^2 on conformally flat manifolds with vanishing scalar curvature in even dimension $n \geq 6$ is of Huygens type too and to find new classes of Lorentzian spin manifolds of even dimension $n \geq 6$ on which D^2 (or a shift of it) is of Huygens type.

Finally we consider the square of twisted Dirac operators on four-dimensional manifolds. We prove that the Huygens property of these operators implies that the coupling connection is flat and that the manifold belongs to one of the three above mentioned classes of Lorentzian manifolds (Theorem 11).

In Section 2 we recall the definition of Huygens operators and describe conditions for normally hyperbolic operators to be of Huygens type. In Section 3 we discuss the Dirac operator on Lorentzian space forms and in Section 4 the Dirac operator on plane wave manifolds. Section 5 deals with coupled Dirac operators on four-dimensional manifolds.

2. Huygens operators

We first recall the definition of normally hyperbolic operators and the properties of their fundamental solutions.

Definition 1. Let M^n be a smooth n-dimensional manifold, (E, p, M) a real or complex vector bundle over M. A differential operator $P : \Gamma(E) \longrightarrow \Gamma(E)$ of second order on E is called normally hyperbolic if there exists a Lorentzian metric such that the principal symbol $\sigma(P)$ of P is given by $\sigma(P)_x(\xi) = -g_x(\xi, \xi) Id_{E_x}$, where $x \in M$ and $\xi \in TM^* \setminus 0$.

Remark. In local coordinates on M and a local trivialization of E a normally hyperbolic operator can be expressed in the form

$$P = -\sum_{i,j=1}^{n} g^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{k=1}^{n} A^k(x) \frac{\partial}{\partial x_k} + B(x),$$

where (g^{ij}) is the inverse matrix of the coefficients of the Lorentzian metric.

Let (M^n, g) be an n-dimensional oriented Lorentzian manifold and let us denote by P the $\mathrm{SO}(n, 1)$-principal bundle of all positive oriented orthonormal frames. A spinor structure of (M^n, g) is a reduction (Q, f) of P with respect to the double covering of the special orthogonal group $\mathrm{SO}(n, 1)$ by the spin group $\mathrm{Spin}(n, 1)$. A Lorentzian spin manifold is an oriented Lorentzian manifold with a fixed spinor structure (Q, f). Let us denote by $\Delta_{n,1}$ the spinor representation of $\mathrm{Spin}(n, 1)$. The complex vector bundle $S := Q \times_{\mathrm{Spin}(n,1)} \Delta_{n,1}$ associated to the principal bundle Q is called spinor bundle of the spin manifold (M^n, g). It is of complex dimension $N = 2^{[n/2]}$. The Levi-Civita connection of (M^n, g) defines a covariant derivative ∇^S in the spinor bundle S given by the formula

$$\nabla^S_X \varphi = X(\varphi) + \frac{1}{2} \sum_{k<l} \varepsilon_k \varepsilon_l \omega_{kl}(X) s_k \cdot s_l \cdot \varphi,$$

where $(s_1, \dots, s_n)$ is a local orthonormal frame, $\varepsilon_j = g(s_j, s_j) = \pm 1$, $\omega_{kl} = g(\nabla s_k, s_l)$ are the connection forms of the Levi-Civita connection ∇ with respect to $(s_1, \dots, s_n)$ and $\cdot$ denotes the Clifford multiplication. Then the Dirac operator of the spin manifold (M^n, g) is defined as the composition of the spinor derivative with the Clifford multiplication μ

$$D : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \stackrel{g}{=} \Gamma(TM \otimes S) \xrightarrow{\mu} \Gamma(S).$$

In local coordinates D can be expressed by

$$D\varphi = \sum_{i=1}^{n} \varepsilon_i s_i \cdot \nabla^S_{s_i} \varphi.$$

Using the spinor calculus it is easy to see that the square D^2 of the Dirac operator is normally hyperbolic. (For details see [Bau81] or [LM89].)

Definition 2. An open subset Ω of M^n is called a geodesically normal domain if it is a normal neighbourhood for each of its points. In particular, for any $x, y \in \Omega$ there exists a unique geodesic $c : [0, 1] \longmapsto \Omega$ with $c(0) = x, c(1) = y$. Let $\sigma : \Omega \times \Omega \longrightarrow \mathbb{R}$ be the quadratic geodesic distance function defined by $\sigma(x, y) := g_{c(t)}(c'(t), c'(t))$, where $c : [0, 1] \to \Omega$ is the unique geodesic in Ω joining x with y.

Let $\Omega \subset M^n$ be a time oriented geodesically normal domain and let x_0 be in Ω. Then

$$\mathcal{J}_+^\Omega(x_0) := \{x \in \Omega \mid \text{the unique geodesic in } \Omega \text{ from } x_0 \text{ to } x \text{ is causal and future oriented}\}$$

is called the future of $x_0 \in \Omega$ and

$$\mathcal{C}_+^\Omega(x_0) := \partial\mathcal{J}_+^\Omega(x_0) = \{x \in \Omega \mid \text{the unique geodesic in } \Omega \text{ from } x_0 \text{ to } x \text{ is light like and future oriented}\}$$

is called the future light cone of $x_0 \in \Omega$. Similarly, we define the past $\mathcal{J}_-^\Omega(x_0)$ and the past light cone $\mathcal{C}_-^\Omega(x_0)$, where now the geodesic is past oriented.

Let (E, p, M) be a vector bundle over M, $\Omega \subset M$ a domain, $x \in \Omega$ and V a vector space. By $\mathcal{D}'(\Omega, E^*; V)$ we denote the space of distributions on $E^*|_\Omega$ with values in V

$$\mathcal{D}'(\Omega, E^*; V) := \{T : \Gamma_0(\Omega, E^*) \longrightarrow V \mid T \text{ linear and continuous}\}.$$

Each differential operator P on E extends to $\mathcal{D}'(\Omega, E^*; V)$ in the following way. If P^* denotes the dual operator on E^*, defined by

$$\int_M \psi(P\varphi)\mu = \int_M (P^*\psi)(\varphi)\mu, \quad \psi \in \Gamma_0(E^*), \quad \varphi \in \Gamma_0(E),$$

then for each distribution $T \in \mathcal{D}'(\Omega, E^*; V)$ the distribution PT is defined by

$$\langle PT, u\rangle := \langle T, P^*u\rangle, \quad u \in \Gamma_0(\Omega, E^*).$$

A distribution $G \in \mathcal{D}'(\Omega, E^*; E_x^*)$ is called fundamental solution of P with respect to (Ω, x) if $PG = \delta_x^{E^*}$, where $\delta_x^{E^*}$ is the Dirac distribution of E_Ω^* centred at x: $\delta_x^{E^*}(u) := u(x)$.

In general there exists no fundamental solution of a normally hyperbolic operator P with respect to (Ω, x). One has to restrict oneself to a certain class of domains Ω, the so-called causal domains.

Definition 3. A domain $\Omega_0 \subset M$ is called causal domain if:
1. Ω_0 is contained in a time and space oriented geodesically normal domain Ω and
2. $\mathcal{J}_+^\Omega(x) \cap \mathcal{J}_-^\Omega(y)$ is compact (or empty) and contained in Ω_0 for all $x, y \in \Omega_0$.

Proposition 1 ([Fri75, Theorem 4.4.1]). *Each Lorentzian manifold can be covered by causal domains.*

Definition 4. A subset $A \subset \Omega_0$ of a causal domain Ω_0 is called past compact (or future compact) if $A \cap \mathcal{J}_-^{\Omega_0}(x)$ (or $A \cap \mathcal{J}_+^{\Omega_0}(x)$) is compact (or empty) for all $x \in \Omega_0$.

Notation. $\mathcal{D}'_+(\Omega_0, E^*; E_x^*) \subset \mathcal{D}'(\Omega_0, E^*; E_x^*)$ denotes the subset of all distributions with past compact support. $\mathcal{D}'_-(\Omega_0, E^*; E_x^*) \subset \mathcal{D}'(\Omega_0, E^*; E_x^*)$ denotes the subset of all distributions with future compact support.

Theorem 2 ([Gün88, Chap. 3.3]). *Let $P : \Gamma(E) \longrightarrow \Gamma(E)$ be a normally hyperbolic operator, let $\Omega_0 \subset M$ be a causal domain and $x \in \Omega_0$. Then there exists exactly one fundamental solution $G_+^{\Omega_0}(x) \in \mathcal{D}'_+(\Omega_0, E^*; E_x^*)$ of P with respect to (Ω_0, x) with past compact support* (forward fundamental solution) *and there exists exactly one fundamental solution $G_-^{\Omega_0}(x) \in \mathcal{D}'_-(\Omega_0, E^*; E_x^*)$ of P with respect to (Ω_0, x) with future compact support* (backward fundamental solution).

The support and singular support of these fundamental solutions satisfy

$$\mathrm{supp} G_\pm^{\Omega_0}(x) \subset \mathcal{J}_\pm^{\Omega_0}(x), \qquad \mathrm{singsupp} G_\pm^{\Omega_0}(x) \subset \mathcal{C}_\pm^{\Omega_0}(x).$$

Now we can define the notion of a Huygens operator.

Definition 5. P is called a Huygens operator or an operator of Huygens type if there exists a covering $\mathcal{U}$ of M by causal domains such that for each causal domain $\Omega_0 \in \mathcal{U}$ and each $x \in \Omega_0$ the forward and backward fundamental solution $G_\pm^{\Omega_0}(x)$ of P with respect to (Ω_0, x) is supported in the light cone $\mathcal{C}_\pm^{\Omega_0}(x)$.

From the structure of the fundamental solutions it is easy to see that a normally hyperbolic operator on a manifold of odd dimension or dimension 2 can never be of Huygens type (see [Gün88, Chap. 3.3]).

Analytic properties of a Huygens operator resulting from the special property of its fundamental solutions can be found in [Gün88, Chap. 4] We will make use of the so-called Hadamard criterion, which says that an operator P is of Huygens type if a certain Hadamard coefficients associated to P vanishes. To make this statement more precise, we will now recall the definition of the Hadamard coefficients of a normally hyperbolic operator.

Let ∇ be a covariant derivative on a vector bundle E over a Lorentzian manifold (M, g). Let us denote by $\nabla^{T^*M\otimes E}$ the covariant derivative defined by the Levi-Civita connection of (M, g) and ∇. The operator

$$\triangle^\nabla := -\mathrm{trace}_g(\nabla^{T^*M\otimes E} \circ \nabla)$$

is called *Bochner–Laplace operator defined by* ∇.

Proposition 2. *Let $P : \Gamma(E) \longrightarrow \Gamma(E)$ be a normally hyperbolic operator on E and let g be the Lorentzian metric given by the principal symbol of P. Then there exists a uniquely* *determined covariant derivative* $\nabla^P : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$ *and a homomorphism* $H_P \in \Gamma(\mathrm{Hom}(E, E))$ *such that*

$$P := \Delta^{\nabla^P} + H_P\ . \tag{1}$$

(Eq. 1) is called Weitzenböck formula for P.)

For two vector bundles E and F over M^n we denote by $E \boxtimes F$ the external tensor product of these bundles over $M \times M$:

$$E \boxtimes F \ := pr_1^* E \otimes pr_2^* F.$$

We often identify the fibre of $E^* \boxtimes E$ over a point $(x, y) \in M \times M$ with the set of homomorphisms $\mathrm{Hom}(E_x, E_y)$. Let $\Omega \subset M$ be a geodesically normal domain in M and let $\sigma : \Omega \times \Omega \longrightarrow \mathbb{R}$ be the quadratic geodesic distance function on Ω. The function $\sigma_x : \Omega \longrightarrow \mathbb{R}$ is given by $\sigma_x(y) := \sigma(x, y)$. The function $m \in C^\infty(\Omega \times \Omega)$

$$m(x, \cdot) := -\tfrac{1}{2}\Delta\sigma_x - n$$

is called *divergence measure* of Ω. By $\tau \in C^\infty(\Omega \times \Omega)$ we denote the function

$$\tau(x, y) := \exp\left\{\frac{1}{2}\int_0^1 \frac{m(x, \gamma(s))}{s}\, \mathrm{d}s\right\},$$

where $\gamma : [0, 1] \longrightarrow \Omega$ denotes the unique geodesic in Ω joining x and y.

Proposition 3. *Let* $P : \Gamma(E) \longrightarrow \Gamma(E)$ *be a normally hyperbolic operator and let* $\Omega \subset M$ *be a geodesically normal domain. Then there exists a uniquely determined sequence of sections* $U_k \in \Gamma(\Omega \times \Omega, E^* \boxtimes E)$, $k = 0, 1, 2, \ldots$, *such that the following differential equations and initial conditions are satisfied on* Ω *for all* $x \in \Omega$:

$$\nabla^P_{\mathrm{grad}\,\sigma_x} U_k(x, \cdot) + (m(x, \cdot) + 2k)U_k(x, \cdot) = -P(U_{k-1}(x, \cdot)),$$
$$U_{-1} = 0, \quad U_0(x, x) = Id_{E_x}.$$

Let us denote by $\mathcal{P}(x, y) \in \mathrm{Hom}(E_x, E_y)$ *the parallel displacement along the geodesic* γ *joining* x *and* y *in* Ω. *Then the sections* U_k *satisfy*

$$U_0(x, y) = \frac{1}{\tau(x, y)}\mathcal{P}(x, y)$$

$$U_k(x, y) = -\frac{1}{2\tau(x, y)}\int_0^1 t^{k-1}\tau(x, \gamma(t))\mathcal{P}(\gamma(t), y)P(U_{k-1}(x, \cdot))(\gamma(t))\, \mathrm{d}t,\ k \geq 1.$$

All differentiations refer to the second component.

The sections $U_k \in \Gamma(\Omega \times \Omega, E^* \boxtimes E), k = 0, 1, 2, \ldots$, are called Hadamard coefficients of P over Ω. Then the Hadamard criterion says:

Theorem 3 ((Hadamard Criterion)). *Let $P : \Gamma(E) \longrightarrow \Gamma(E)$ be a normally hyperbolic operator over a manifold M of even dimension $n \geq 4$. Then P is a Huygens operator if and only if there exists a covering $\mathcal{U}$ by causal domains such that the Hadamard coefficient $U_{(n-2)/2}$ of P satisfies*

$$U_{(n-2)/2}(x, y) = 0$$

for all $\Omega_0 \in \mathcal{U}, x \in \Omega_0, y \in C^{\Omega_0}(x)$.

Another method to find Huygens operators is founded on the conformal gauge invariance of the Huygens property. Let M^n be a manifold of even dimension $n \geq 4$ and let E be a vector bundle over M^n. To each normally hyperbolic operator $P : \Gamma(E) \longrightarrow \Gamma(E)$ is assigned a sequence $I_k(P) \in \Gamma(S^k(T^*M) \otimes \mathrm{Hom}(E, E))$, $k = 0, 1, 2, \ldots$, of symmetric trace free conformal gauge invariants of weight $\omega = 1 - n/2$, the so-called moments of P of order k (see [Gün88, Chap. 6]). If P is a Huygens operator, then all moments $I_k(P)$, $k = 0, 1, 2, \ldots$, vanish. Moreover, if M and P are analytic, the vanishing of all moments imply the Huygens property for P. In dimension $n = 4$ there are explicit formulas which express the moments of order ≤ 4 in terms of the curvature of the manifold (M^4, g) and the curvature of the covariant derivative ∇^P associated to P by its Weitzenböck formula (see [Gün88, Chap. 7, Table II]). Using these formulas for the first three moments one obtains:

Theorem 4. *Let $P : \Gamma(E) \longrightarrow \Gamma(E)$ be a normally hyperbolic operator of Huygens type on a four-dimensional manifold M^4. Let g be the Lorentzian metric defined by P, R the scalar curvature of (M^4, g), ∇^P the covariant derivative and H_P the homomorphism on E associated to P by its Weitzenböck formula. Then the following conditions are satisfied:*
1. *The Cotton invariant $C_P := H_P - \frac{1}{6}R$ of P vanishes.*
2. *∇^P is a Yang–Mills connection: for the curvature F^P of ∇^P, $\delta F^P = 0$ holds.*
3. *The Bach tensor $\mathcal{B}$ of (M^4, g) equals a multiple of the "energy impulse tensor" of F^P:*

$$\mathcal{B} \otimes Id_E = -5\big(Q_2(F^P, F^P) - \tfrac{1}{4} g \otimes \|F^P\|^2\big).$$

3. The Dirac operator of Lorentzian space forms

Let (M^n, g) be a Lorentzian spin manifold of constant sectional curvature K and even dimension $n \geq 4$. Let us denote by $D : \Gamma(S) \longrightarrow \Gamma(S)$ the Dirac operator of (M^n, g). In this section we will prove that the operator $D^2 - K : \Gamma(S) \longrightarrow \Gamma(S)$ is of Huygens type.

The proof is based on the fact that the spinor bundle over a simply connected semi-Riemannian space form can be trivialized by Killing spinors and on the fact that the Yamabe operator of a conformally flat Lorentzian manifold is of Huygens type (see [Gün88, Chap. 6.3]).

Definition 6. Let (M^n, g) be a Lorentzian spin manifold with spinor bundle S. A spinor field $\varphi \in \Gamma(S)$ is called a Killing spinor to the Killing number $\lambda \in \mathbb{C}$ if

$$\nabla^S_X \varphi = \lambda X \cdot \varphi$$

for all vector fields X on M.

Let us denote by M^n_K the complete simply connected Lorentzian manifold of constant sectional curvature K. Since M^n_K is parallelizable, it is orientable and has exactly one spinor structure (the trivial spinor structure). Let S be the spinor bundle of M^n_K.

Proposition 4 ([CGLS86]). *The spinor bundle S on M^n_K can be trivialized by Killing spinors to the Killing number $\mu \in \mathbb{C}$, where $\mu^2 = \frac{1}{4}K$.*

Proof. Let $\mu \in \mathbb{C}$ be a complex number with $\mu^2 = \frac{1}{4}K$ and let denote ∇^μ the covariant derivative

$$\nabla^\mu_X := \nabla^S_X - \mu X$$

on S. A Killing spinor to the Killing number μ is parallel with respect to ∇^μ. Since M^n_K is simply connected it is enough to prove that the curvature F^μ of ∇^μ vanishes on the bundle S. Now, it is easy to check that

$$F^\mu(X, Y) = \nabla^\mu_X \nabla^\mu_Y - \nabla^\mu_Y \nabla^\mu_X - \nabla^\mu_{[X,Y]} = F^S(X, Y) + \mu^2(X \cdot Y - Y \cdot X). \tag{2}$$

The curvature F^S of the spinor derivative ∇^S is given by

$$F^S(X, Y) = \frac{1}{4} \sum_{ij} \varepsilon_i \varepsilon_j \mathcal{R}(X, Y, s_i, s_j) s_i \cdot s_j \cdot,$$

where $\mathcal{R}$ denotes the curvature tensor of the basis manifold and $(s_1, \dots, s_n)$ is a local orthonormal basis. The curvature tensor of a manifold of constant sectional curvature K is given by

$$\mathcal{R}(X, Y, V, W) = K\{g(X, W)g(Y, V) - g(X, V)g(Y, W)\}.$$

Hence on M^n_K we have, for the curvature of the spinor derivative,

$$F^S(X, Y) = \tfrac{1}{4}K(Y \cdot X - X \cdot Y).$$

Since $\mu^2 = \frac{1}{4}K$ from (2) it follows that $F^\mu \equiv 0$ on S. □

Helgason [Hel94, Chap. 5.5.4] and Schimming and Schlichtkrull [SS94] proved the following theorem for the Laplace operator of a Lorentzian space form.

Theorem 5. *Let (M^n, g) be a Lorentzian manifold of constant sectional curvature K and even dimension $n \geq 4$. Then for each $m = 3, 5, \dots, n-1$ the shifted Laplace operator*

$$L_m := \Delta_0 + K\,(n - m)(m - 1)$$

is of Huygens type.

This can be used to prove the following theorem.

Theorem 6. *Let (M^n, g) be a Lorentzian spin manifold of constant sectional curvature K and even dimension $n \geq 4$ and denote by D the Dirac operator of (M^n, g). Let $\mu \in \mathbb{C}$ be a complex number with $\mu^2 = \frac{1}{4}K$. Then for each $m = 3, 5, \ldots, n-1$ the operators*

$$P_{m,-} := (D-\mu)^2 - \tfrac{1}{4}K\{(n-1)^2 - 4(n-m)(m-1)\}$$

and

$$P_{m,+} := (D+\mu)^2 - \tfrac{1}{4}K\{(n-1)^2 - 4(n-m)(m-1)\}$$

are of Huygens type.

Proof. Let $\lambda \in \mathbb{C}$ be a complex number. The Weitzenböck formula of $(D+\lambda)^2$ is

$$(D+\lambda)^2 = \Delta^\lambda + \tfrac{1}{4}n(n-1)K + (1-n)\lambda^2, \tag{3}$$

where Δ^λ is the Bochner–Laplace operator of the covariant derivative ∇^λ defined by $\nabla^\lambda_X := \nabla^S_X - \lambda X$,

$$\Delta^\lambda = -\sum_{i=1}^{n} \varepsilon_i (\nabla^\lambda_{s_i} \nabla^\lambda_{s_i} + \operatorname{div}(s_i)\nabla^\lambda_{s_i}).$$

For a function $f \in C^\infty(M)$ and a spinor field $\psi \in \Gamma(S)$ we have

$$\Delta^\lambda(f\psi) = \Delta_0(f)\psi + f\Delta^\lambda\psi - 2\nabla^\lambda_{\operatorname{grad}(f)}\psi. \tag{4}$$

Hence, if φ is a Killing spinor to the Killing number μ with $\mu^2 = \frac{1}{4}K$, then from (3) and (4) it follows that

$$(D+\mu)^2(f\varphi) = \Delta_0(f)\varphi + \tfrac{1}{4}K(n-1)^2 f\varphi. \tag{5}$$

Since the Huygens property is a local one, it is enough to prove the theorem for the simply connected manifold M^n_K. According to Proposition 4 there exists a basis of Killing spinors $\varphi_1, \ldots, \varphi_N$ to the Killing number μ in the spinor bundle S of M^n_K. Hence the space of spinor fields $\Gamma(M^n_K, S)$ on M^n_K can be identified with that of smooth functions $C^\infty(M^n_K, \mathbb{C}^N)$ by assigning to each spinor field $\varphi = \sum_{\varepsilon=1}^{N} f_\varepsilon \varphi_\varepsilon$ the function $(f_1, \ldots, f_N)$. Using this identification according to (5) the operator $(D+\mu)^2$ corresponds to the operator $\Delta_0 + K/4(n-1)^2$. Then from Theorem 5 it follows that for each $m = 3, 5, \ldots, n-1$ the operator

$$P_{m,+} = (D+\mu)^2 - \tfrac{1}{4}K\{(n-1)^2 - 4(n-m)(m-1)\}$$

acting on $\Gamma(M^n_K, S)$ is of Huygens type. If we trivialize the spinor bundle of M^n_K using Killing spinors to the Killing number $-\mu$ we obtain in the same manner the result for the operator $P_{m,-} = (D-\mu)^2 - \frac{1}{4}K\{(n-1)^2 - 4(n-m)(m-1)\}$. □

Now, we are able to prove the above-mentioned result.

Theorem 7. *Let (M^n, g) be a Lorentzian spin manifold of constant sectional curvature K and even dimension $n \geq 4$ and let us denote by D the Dirac operator of (M^n, g). Then the operator $D^2 - K : \Gamma(S) \longrightarrow \Gamma(S)$ is of Huygens type.*

Proof. If K is zero, the theorem follows from Theorem 6. Hence we assume that $K \neq 0$. Let μ be a complex number with $\mu^2 = \frac{1}{4}K$ and let $m = 2[\frac{1}{4}n] + 1$. According to Theorem 6 the operators $P_{m,-} = (D-\mu)^2 - \mu^2 = (D-2\mu)D$ and $P_{m,+} = (D+\mu)^2 - \mu^2 = (D+2\mu)D$ are Huygens operators. (In the simply connected case, these operators correspond to the Yamabe operator acting on $C^\infty(M^n_K, \mathbb{C}^N)$, if we trivialize the spinor bundle S using Killing spinors to the Killing number μ and $-\mu$, respectively). Since $(D - 2\mu)D$ and $(D + 2\mu)D$ are of Huygens type, there exists a covering $\mathcal{U}$ by causal domains such that for all $\Omega_0 \in \mathcal{U}$ and $x \in \Omega_0$ the forward and backward fundamental solutions $G^{\Omega_0}_{1,\pm}(x)$ and $G^{\Omega_0}_{2,\pm}(x)$ of $(D - 2\mu)D$ and $(D + 2\mu)D$, respectively, are supported in the light cone $C^{\Omega_0}_{\pm}(x)$. Consider the distributions $H^{\Omega_0}_{1,\pm}(x) := DG_{1,\pm}(x)$ and $H^{\Omega_0}_{2,\pm}(x) := DG_{2,\pm}(x)$. Then

$$(D + 2\mu)(D - 2\mu)H^{\Omega_0}_{1,\pm}(x) = (D + 2\mu)\delta_x = D\delta_x + 2\mu\delta_x,$$
$$(D - 2\mu)(D + 2\mu)H^{\Omega_0}_{2,\pm}(x) = (D - 2\mu)\delta_x = D\delta_x - 2\mu\delta_x.$$

Hence

$$(D^2 - 4\mu^2)\left(\frac{1}{4\mu}[H^{\Omega_0}_{1,\pm}(x) - H^{\Omega_0}_{2,\pm}(x)]\right) = \delta_x.$$

Therefore, the distribution $E^{\Omega_0}_{\pm}(x) \in \mathcal{D}'_{\pm}(\Omega_0, S^*; S_x)$ defined by

$$E^{\Omega_0}_{\pm}(x) := \frac{1}{4\mu}[H^{\Omega_0}_{1,\pm}(x) - H^{\Omega_0}_{2,\pm}(x)]$$

is the forward resp. backward fundamental solution of the operator $D^2 - K$ with respect to (Ω_0, x). Since

$$\begin{aligned} \operatorname{supp} E^{\Omega_0}_{\pm}(x) &\subset \operatorname{supp} H^{\Omega_0}_{1,\pm}(x) \cup \operatorname{supp} H^{\Omega_0}_{2,\pm}(x) \\ &\subset \operatorname{supp} G^{\Omega_0}_{1,\pm}(x) \cup \operatorname{supp} G^{\Omega_0}_{2,\pm}(x) \\ &\subset C^{\Omega_0}_{\pm}(x), \end{aligned}$$

$D^2 - K$ is a Huygens operator. □

4. The Dirac operator on plane wave manifolds

Already for a long time it has been known that the Laplace–Beltrami operator on a plane wave manifold of even dimension $n \geq 4$ and the Hodge–Laplace operators on forms of a plane wave manifold of even dimension $n \geq 6$ are of Huygens type (see

[Gün65,Sch71,Gün88]). In this section we prove that the square of the Dirac operator on plane wave spin manifolds of even dimension $n \geq 4$ is a Huygens operator.

Definition 7. A Lorentzian manifold (M^n, g) is called a plane wave manifold if the following conditions are satisfied:
1. There exists an isotropic parallel vector field T on M.
2. For the curvature tensor $\mathcal{R}$ of (M, g)

$$\mathrm{trace}_{(3,5),(4,6)}\mathcal{R} \otimes \mathcal{R} = 0$$

holds, where $\mathrm{trace}_{(i,j)}B$ denotes the trace of the tensor field B in the ith and jth component with respect to g.
3. $\mathcal{R}$ is quasi recurrent with T, i.e. there exists a (4,0)-tensor field $\mathcal{R}_1$ such that

$$\nabla\mathcal{R} = T^{\flat} \otimes \mathcal{R}_1,$$

where $T^{\flat}$ denotes the 1-form dual to T with respect to g.

A plane wave manifold is foliated by submanifolds of codimension 1, the integral curves of T are isotropic geodesics running in the leaves of the foliation and M is locally symmetric along the leaves: $\nabla_X\mathcal{R} = 0$ for all vectors X tangent to the leaves of the foliation. The scalar curvature of a plane wave manifold is zero.

The geometry of plane wave manifolds was studied by Schimming in [Sch74].

Theorem 8 ([Sch74]). *A Lorentzian manifold (M^n, g) is a plane wave manifold with the isotropic parallel vector field T if and only if for each point $x \in M$ there exists a coordinate neighbourhood $(U, (x^1, \ldots, x^n))$ such that the metric g has the form*

$$g_{|U} = 2\,\mathrm{d}x^1\,\mathrm{d}x^2 + a_{\alpha\beta}(x^1)\,\mathrm{d}x^{\alpha}\,\mathrm{d}x^{\beta}, \quad 3 \leq \alpha, \beta \leq n,$$

where $(a_{\alpha\beta})$ is a positive definite matrix, depending only on x^1 and the vector field T is given by $T_{|U} = \partial/\partial x^2$.

4.1. Some geometry of standard plane wave manifolds

In this section we consider the following "standard" plane wave manifold (M, g): Let $M := I \times \mathbb{R}^{n-1}$ be the product of an open interval I with the $\mathbb{R}^{n-1}$ and g be the Lorentzian metric

$$g = 2\,\mathrm{d}x_1\,\mathrm{d}x_2 + \,\mathrm{d}\widetilde{x}A(x_1)\,\mathrm{d}\widetilde{x}^{\,t},$$

where $x = (x_1, \ldots, x_n) \in M, \widetilde{x} = (x_3, \ldots, x_n)$ and $A(x_1)$ denotes a positive definite $(n-2) \times (n-2)$-matrix, depending smoothly on x_1. Then $T := \partial/\partial x_2$ is the isotropic, parallel vector field occurring in the definition of a plane wave manifold. We will explain some geometric properties of standard plane wave manifolds which we need to calculate the Hadamard coefficients of the square of the Dirac operator.

With small letters $i, j, k, l, \ldots$ we will denote indices running from $1, \ldots, n$. With capital letters $I, J, K, L, \ldots$ we will denote indices running from $3, \ldots, n$. $(\partial_1, \ldots, \partial_n)$ is the canonical basis with respect to the coordinates $(x_1, \ldots, x_n)$.

We will use the following denotations:

Definition 8. Let A be the matrix occurring in the definition of the metric g. Then V and W are the symmetric matrices:

$$V(q_1, q_2) := \int_0^1 A^{-1}(q_1 + t(q_2 - q_1))\,\mathrm{d}t, \qquad W(q_1, q_2) := (V(q_1, q_2))^{-1}.$$

By a simple calculation it follows:

Lemma 1. *The only non-vanishing Christoffel symbols of (M, g) with respect to the coordinates $(x_1, \ldots, x_n)$ are*

$$\Gamma_{1I}^{K} = \Gamma_{I1}^{K} = \tfrac{1}{2}(A^{-1}\dot{A})_{KI}, \quad I, K = 3, \ldots, n,$$
$$\Gamma_{IJ}^{2} = -\tfrac{1}{2}\dot{A}_{IJ}, \quad I, J, = 3, \ldots, n,$$

where $\dot{A}$ denotes the derivative of A with respect to x_1.

The geodesics of the standard plane wave manifold (M, g) are described in the following proposition.

Proposition 5. *Let y, z be two points of M. Then there is exactly one geodesic $\gamma = (\gamma_1, \gamma_2, \widetilde{\gamma}) : [0, 1] \longrightarrow M$ joining y with z.*

This geodesic is given by

$$\gamma_1(s) = y_1 + (z_1 - y_1)s,$$
$$\gamma_2(s) = \begin{cases} y_2 + (z_2 - y_2)s + \dfrac{s}{2(z_1 - y_1)}(\widetilde{z} - \widetilde{y})[W - WV(s)W](\widetilde{z} - \widetilde{y})^t \\ \quad \text{if } z_1 \neq y_1, \\ y_2 + (z_2 - y_2)s + \dfrac{1}{4}(s^2 - s)(\widetilde{z} - \widetilde{y})\dot{A}(y_1)(\widetilde{z} - \widetilde{y})^t \quad \text{if } z_1 = y_1, \end{cases}$$
$$\widetilde{\gamma}(s)^t = \widetilde{y}^{\,t} + sV(s)W(\widetilde{z} - \widetilde{y})^t,$$

where $V(s)$ and W are the matrices

$$V(s) := V(y_1, \gamma_1(s)), \qquad W := W(y_1, z_1).$$

In particular, M is a geodesically normal domain.

Proof. Let $\delta(s) = (\delta_1(s), \delta_2(s), \widetilde{\delta}(s))$ be the geodesic with $\delta(0) = y = (y_1, y_2, \widetilde{y})$, $\delta'(0) = v = (v_1, v_2, \widetilde{v})$. Using Lemma 1, the geodesic equations

$$\delta_k''(s) + \sum_{ij} \delta_i'(s)\delta_j'(s)\Gamma_{ij}^{k}(\delta(s)) = 0$$

result

$$\delta_1''(s) = 0, \tag{6}$$
$$\delta_2''(s) = \tfrac{1}{2}\widetilde{\delta}'(s)\dot{A}(\delta_1(s))\widetilde{\delta}'(s)^t, \tag{7}$$
$$\widetilde{\delta}''(s)^t = -\delta_1'(s)A^{-1}(\delta_1(s))\dot{A}(\delta_1(s))\widetilde{\delta}'(s)^t. \tag{8}$$

Hence,

$$\delta_1(s) = y_1 + v_1 s. \tag{9}$$

Let $\mathcal{A}(s) := A(\delta_1(s))$. Using $\mathcal{A}'(s) = \dot{A}(\delta_1(s))\delta_1'(s)$ we obtain from (8)

$$\mathcal{A}(s)\widetilde{\delta}''(s)^t + \mathcal{A}'(s)\widetilde{\delta}'(s)^t = (\mathcal{A}\widetilde{\delta}'^t)'(s) = 0.$$

From the initial data it follows

$$\widetilde{\delta}(s)^t = \left(\int_0^s \mathcal{A}^{-1}(t)\,\mathrm{d}t\right) A(y_1)\widetilde{v}^{\,t} + \widetilde{y}^{\,t} = \mathcal{V}(s)A(y_1)\widetilde{v}^{\,t}s + \widetilde{y}^{\,t}, \tag{10}$$

where $\mathcal{V}(s) := V(y_1, \delta_1(s))$.

Now, let $v_1 \neq 0$. Then (7) and (10) yield

$$\delta_2''(s) = -\frac{1}{2v_1}\widetilde{v}A(y_1)(\mathcal{A}^{-1})'(s)A(y_1)\widetilde{v}^{\,t}.$$

Hence,

$$\delta_2(s) = y_2 + v_2 s + \frac{s}{2v_1}\widetilde{v}[A(y_1) - A(y_1)\mathcal{V}(s)A(y_1)]\widetilde{v}^{\,t}. \tag{11}$$

In case $v_1 = 0$, from $\delta_1(s) = y_1$ it follow $\mathcal{A}(s) \equiv A(y_1)$ and $\mathcal{V}(s) \equiv A^{-1}(y_1)$. Therefore, (10) gives

$$\widetilde{\delta}(s) = \widetilde{y} + \widetilde{v}s$$

and (7) results

$$\delta_2(s) = y_2 + v_2 s + \tfrac{1}{4}\widetilde{v}\dot{A}(y_1)\widetilde{v}^{\,t}s^2. \tag{12}$$

Now, let $\gamma : [0, 1] \longrightarrow M$ be a geodesic in M joining y with z and let $v \in T_yM$ be the vector $v = \gamma'(0)$. Then with $\gamma(s) = \delta(s)$ formulas (9)–(12) show that v is uniquely determined by $z = \delta(1)$. We obtain

$$v_1 = z_1 - y_1, \tag{13}$$
$$v_2 = \begin{cases} z_2 - y_2 - \dfrac{1}{2(z_1 - y_1)}(\widetilde{z} - \widetilde{y})[WA^{-1}(y_1)W - W](\widetilde{z} - \widetilde{y})^t & \text{if } z_1 \neq y_1, \\ z_2 - y_2 - \dfrac{1}{4}(\widetilde{z} - \widetilde{y})\dot{A}(y_1)(\widetilde{z} - \widetilde{y})^t & \text{if } z_1 = y_1, \end{cases} \tag{14}$$
$$\widetilde{v}^{\,t} = A^{-1}(y_1)W(\widetilde{z} - \widetilde{y})^t, \tag{15}$$

Inserting this in (9)–(12) the proposition is proved. □

Now, we can calculate the quadratic geodesic distance function and the divergence measure of (M, g):

Proposition 6. *The quadratic geodesic distance function $\sigma : M \times M \longrightarrow \mathbb{R}$ of the standard plane wave manifold (M, g) is given by*

$$\sigma(y, z) = 2(z_1 - y_1)(z_2 - y_2) + (\widetilde{z} - \widetilde{y})W(y_1, z_1)(\widetilde{z} - \widetilde{y})^t.$$

Proof. Let $\gamma : [0, 1] \longrightarrow M$ be the geodesic joining y with z and let $v = \gamma'(0)$. By definition

$$\sigma(y, z) = g(\gamma'(s), \gamma'(s)) \equiv g(\gamma'(0), \gamma'(0)) = 2v_1 v_2 + \widetilde{v}A(y_1)\widetilde{v}^{\,t}.$$

Inserting (13)–(15) and regard $W(y_1, y_1) = A(y_1)$ we obtain the assertion. □

Proposition 7. *The divergence measure of the standard plane wave manifold (M^n, g) is given by*

$$m(y, z) = -(z_1 - y_1)\left(\ln \frac{w_{y_1}}{\sqrt{a}}\right)'(z_1),$$

where $w_{y_1}(z_1) := w(y_1, z_1) := \det[W(y_1, z_1)]$, $a(z_1) := \det[A(z_1)]$.

Proof. Let $\sigma_y(z) := \sigma(y, z)$ and $W_{y_1}(z_1) = W(y_1, z_1)$. By definition the divergence measure $m(y, z)$ is

$$m(y, \cdot) = -\tfrac{1}{2}\Delta(\sigma_y) - n.$$

An easy calculation shows that the Laplacian Δ of a function $u \in C^\infty(M)$ is

$$\Delta u(x) = -2\frac{\partial^2 u}{\partial x_1 \partial x_2}(x) - A(x_1)^{IJ}\frac{\partial^2 u}{\partial x_I \partial x_J}(x) - (\ln \sqrt{a})'(x_1)\frac{\partial u}{\partial x_2}(x)$$

where $(A^{IJ}(x_1))$ denote the elements of the inverse matrix $A^{-1}(x_1)$. Using Proposition 6 we obtain

$$\Delta(\sigma_y)(z) = -4 - 2\,\mathrm{trace}\,(A^{-1}W_{y_1})(z_1) - 2(z_1 - y_1)(\ln \sqrt{a})'(z_1).$$

Since $\mathrm{trace}(W_{y_1}^{-1}W_{y_1}) = n - 2$ we can deduce

$$m(y, z) = \mathrm{trace}\,[(A^{-1} - W_{y_1}^{-1})W_{y_1}](z_1) + (z_1 - y_1)(\ln \sqrt{a})'(z_1).$$

From

$$W^{-1}(y_1, z_1) = \int_0^1 A^{-1}(y_1 + t(z_1 - y_1))\,\mathrm{d}t$$

it follows

$$A^{-1}(z_1) - W_{y_1}^{-1}(z_1) = (W_{y_1}^{-1})'(z_1)\,(z_1 - y_1).$$

Hence,

$$m(y,z) = (z_1 - y_1)\,\mathrm{trace}[(W_{y_1}^{-1})' W_{y_1}](z_1) + (z_1 - y_1)(\ln\sqrt{a})'(z_1). \tag{16}$$

For a symmetric positive definite matrix $U(t)$

$$(\ln\det(U))' = \mathrm{trace}\,[U^{-1}U'] = -\mathrm{trace}\,[(U^{-1})'U]$$

holds. Therefore (16) gives

$$m(y,z) = -(z_1 - y_1)\left(\ln\left(\frac{w_{y_1}}{\sqrt{a}}\right)\right)'(z_1). \qquad \square$$

Let $\tau \in C^\infty(M \times M)$ be the function

$$\tau(y,z) := \exp\left\{\frac{1}{2}\int_0^1 \frac{m(y,\gamma(s))}{s}\,\mathrm{d}s\right\}$$

where $\gamma : [0,1] \longrightarrow M$ denotes the unique geodesic joining y with z.

Proposition 8. *The function τ is given by*

$$\tau(y,z) = \frac{\sqrt[4]{a(y_1)a(z_1)}}{\sqrt{w(y_1,z_1)}}.$$

Proof. From Proposition 5 we know

$$\gamma_1(s) = y_1 + (z_1 - y_1)s.$$

Using Proposition 7 we obtain

$$m(y,\gamma(s)) = -(z_1 - y_1)s\left(\ln\frac{w_{y_1}}{\sqrt{a}}\right)'(\gamma_1(s)) = -s\frac{\mathrm{d}}{\mathrm{d}s}\left(\ln\frac{w(y_1,\gamma_1(s))}{\sqrt{a(\gamma_1(s))}}\right).$$

It follows

$$\tau(y,z) = \exp\left\{\frac{1}{2}\left[\ln\frac{w(y_1,y_1)}{\sqrt{a(y_1)}} - \ln\frac{w(y_1,z_1)}{\sqrt{a(z_1)}}\right]\right\}.$$

Since $w(y_1,y_1) = a(y_1)$ this gives

$$\tau(y,z) = \exp\left\{\frac{1}{2}\ln\frac{\sqrt{a(z_1)a(y_1)}}{w(z_1,y_1)}\right\} = \frac{\sqrt[4]{a(z_1)a(y_1)}}{\sqrt{w(y_1,z_1)}}. \qquad \square$$

We now determine the parallel displacement of the canonical basis vectors $(\partial_1(y), \ldots, \partial_n(y))$ along geodesics starting from a fixed point $y \in M$.

Proposition 9. *Let y and z be two points in M with $y_1 \neq z_1$. Let us denote by $\mathfrak{b} \in \mathbb{R}^{n-2}$ the vector*

$$\mathfrak{b} := \frac{1}{z_1 - y_1}(\tilde{z} - \tilde{y})W(y_1, z_1)$$

and by $P = P(x_1)$ the matrix given by the initial value problem

$$\dot{P} = -\tfrac{1}{2}P\dot{A}A^{-1}, \qquad P(y_1) = E.$$

Then the parallel displacement $\mathfrak{p} : T_yM \longrightarrow T_zM$ along the geodesic joining y with z satisfies

$$\mathfrak{p}(\partial_1(y)) = \partial_1(z) + \tfrac{1}{2}\mathfrak{b}[A^{-1}(y_1)(2P(z_1) - E) - A^{-1}(z_1)]\mathfrak{b}^t\partial_2(z) + (\mathfrak{b}[A^{-1}(z_1) - A^{-1}(y_1)P(z_1)])_K\partial_K(z), \tag{17}$$

$$\mathfrak{p}(\partial_2(y)) = \partial_2(z), \tag{18}$$

$$\mathfrak{p}(\partial_L(y)) = P_{LK}(z_1)\partial_K(z) + ((E - P(z_1))\mathfrak{b}^t)_L\partial_2(z), \quad L = 3, \dots, n. \tag{19}$$

Proof. Let γ be the geodesic joining y with z and let $X(t) = a_j(t)\partial_j(\gamma(t))$ be a vector field parallel along γ. Denote $\mathcal{A}(t) := A(\gamma_1(t))$ and $W = W(y_1, z_1)$. If we insert the results of Lemma 1 and Proposition 5 into the equation

$$a_i'(t) + \sum_{j,k} \gamma_k'(t)a_j(t)\gamma_{kj}^i(t) = 0, \quad i = 1, \dots, n$$

for the parallel displacement, we obtain

$$a_1'(t) = 0, \qquad a_2'(t) = \tfrac{1}{2}\mathfrak{b}(\mathcal{A}^{-1}\mathcal{A}')(t)\widetilde{a}\,{}^t(t),$$
$$\widetilde{a}''(t) = -\tfrac{1}{2}\{(\mathcal{A}^{-1}\mathcal{A}'\widetilde{a}\,{}^t)(t) - a_1(t)(\mathcal{A}^{-1})'(t)\mathfrak{b}^t\},$$

where $\widetilde{a}(t) = (a_3(t), \dots, a_n(t))$. Since $\partial_2 = T$ is the isotropic parallel vector field on the standard plane wave manifold, (18) holds.

Now, let $X(t) = a_{Lj}(t)\partial_j(\gamma(t))$ be the parallel displacement of $\partial_L(y)$. Then the initial condition yields $a_{L1}(t) = 0$. Therefore

$$\widetilde{a}_L''(t) = -\tfrac{1}{2}\mathcal{A}^{-1}\mathcal{A}'\widetilde{a}_L^{\,t}(t), \qquad a_{LK}(0) = \delta_{LK}.$$

This is solved by $a_{LK}(t) := P_{LK}(\gamma_1(t))$. For $a_{L2}(t)$ it follows $a_{L2}'(t) = -\langle \mathfrak{b}, \widetilde{a}_L'(t)\rangle$ and the initial condition gives $a_{L2}(t) = \mathfrak{b}_L - \langle \mathfrak{b}, \widetilde{a}_L(t)\rangle$. This proves (19).

Now, let $X(t) = a_{1j}(t)\partial_j(\gamma(t))$ be the parallel displacement of $\partial_1(y)$. Then $a_{11}(t) = 1$. Therefore

$$\widetilde{a}_1''(t) = -\tfrac{1}{2}\mathcal{A}^{-1}\mathcal{A}'\widetilde{a}_1^{\,t}(t) + \tfrac{1}{2}(\mathcal{A}^{-1})'\mathfrak{b}^t, \qquad \widetilde{a}_1(0) = 0.$$

This initial value problem is solved by the vector

$$\widetilde{a}_1(t) = \mathfrak{b}[\mathcal{A}^{-1}(t) - A^{-1}(y_1)P(\gamma_1(t))].$$

For a_{12} it follows

$$a_{12}'(t) = -\widetilde{a}_1'(t)\cdot\mathfrak{b}^t + \tfrac{1}{2}\mathfrak{b}(\mathcal{A}^{-1})'(t)\mathfrak{b}^t$$

and the initial condition gives

$$a_{12}(t) = \tfrac{1}{2}\mathfrak{b}[-\mathcal{A}^{-1}(t) + A^{-1}(y_1)(2P(\gamma_1(t)) - E)]\mathfrak{b}^t.$$

This proves (17). □

4.2. The Dirac operator of a standard plane wave manifold

In this section (M,g) denotes a standard plane wave manifold as it was defined in Section 4.1. Since M is parallelizable, (M,g) is an oriented Lorentzian manifold with a unique (trivial) spinor structure Q. By S we denote the associated spinor bundle $S = Q \times_{\mathrm{Spin}(n,1)} \Delta_{n,1}$.

Let y be a fixed point in M. We choose a $(n-2)\times(n-2)$-matrix C such that $CA(y_1)C^t = E$. Then

$$s_1(y) := \frac{1}{\sqrt{2}}(\partial_1(y) - \partial_2(y)), \qquad s_2(y) := \frac{1}{\sqrt{2}}(\partial_1(y) + \partial_2(y)),$$
$$s_I(y) := C_{IJ}\partial_J(y), \quad I = 3,\dots,n$$

is an orthonormal basis in T_yM. We denote by $s = (s_1,\dots,s_n)$ the global orthonormal basis on M arising from $(s_1(y),\dots,s_n(y))$ by parallel displacement along geodesics. Let the orientation of M be fixed by this basis s. $\hat{s}$ denotes a lift of s in the spinor structure Q.

Proposition 10. *Let $v_1,\dots,v_N$ be a basis in the spinor module $\Delta_{n,1}$ and let us denote by $\eta_1,\dots,\eta_N \in \Gamma(S)$ the basis sections in the spinor bundle defined by*

$$\eta_\varepsilon(x) := [\hat{s}(x), v_\varepsilon], \quad \varepsilon = 1,\dots,N = 2^{[n/2]}.$$

Then over the open submanifold $\{x \in M | x_1 \neq y_1\}$ the spinor derivative of η_ε is given by

$$\nabla^S_{\partial_1}\eta_\varepsilon = \frac{1}{\sqrt{2}}\sum_I \omega_{1I}(\partial_1)T\cdot s_I\cdot\eta_\varepsilon + \frac{1}{2}\sum_{I<J}\omega_{IJ}(\partial_1)s_I\cdot s_J\cdot\eta_\varepsilon, \tag{20}$$

$$\nabla^S_{\partial_2}\eta_\varepsilon = 0, \tag{21}$$

$$\nabla^S_{\partial_K}\eta_\varepsilon = \frac{1}{\sqrt{2}}\sum_I \omega_{1I}(\partial_K)T\cdot s_I\cdot\eta_\varepsilon, \quad K = 3,\dots,n, \tag{22}$$

where $\omega_{ij} = g(\nabla s_i, s_j)$ are the connection forms of the Levi-Civita connection with respect to the basis $(s_1,\dots,s_n)$. The connection coefficients $\omega_{1I}(\partial_K)$ and $\omega_{IJ}(\partial_1)$ depend only on the first variable x_1.

Proof. The spinor derivative of η_ε is given by

$$\nabla^S_X\eta_\varepsilon = \frac{1}{2}\sum_{i<j}\varepsilon_i\varepsilon_j\omega_{ij}(X)\, s_i\cdot s_j\cdot\eta_\varepsilon. \tag{23}$$

Hence we have to calculate ω_{ij}.

Using Proposition 9 we obtain over $\{x \in M | x_1 \neq y_1\}$

$$s_1 = \frac{1}{\sqrt{2}}[\partial_1 + (\tfrac{1}{2}\mathfrak{b}[A^{-1}(y_1)(2P-E) - A^{-1}]\mathfrak{b}^t - 1)\partial_2 + (\mathfrak{b}[A^{-1} - A^{-1}(y_1)P])_K\partial_K], \tag{24}$$

$$s_2 = \frac{1}{\sqrt{2}}\left[\partial_1 + \left(\frac{1}{2}\mathfrak{b}[A^{-1}(y_1)(2P-E) - A^{-1}]\mathfrak{b}^t + 1\right)\partial_2\right. \tag{25}$$

$$+ (\mathfrak{b}[A^{-1} - A^{-1}(y_1)P])_K \partial_K \Bigg],$$

$$s_I = (CP)_{IK}\partial_K + (C(E-P)\mathfrak{b}^t)_I \partial_2, \tag{26}$$

where $\mathfrak{b}(x)$ is the vector

$$\mathfrak{b}(x) = \frac{1}{x_1 - y_1}(\widetilde{x} - \widetilde{y}) W(y_1, x_1)$$

and $P = P(x_1)$ is the matrix given by

$$\dot{P} = -\tfrac{1}{2} P \dot{A} A^{-1}, \qquad P(y_1) = E.$$

Since by Lemma 1 $\Gamma^j_{2i} = 0$ for all $i, j = 1, \dots, n$ and the coefficients of s_i with respect to $(\partial_1, \dots, \partial_n)$ do not depend on the second variable, it follows

$$\omega_{ij}(\partial_2) = 0, \quad i, j = 1, \dots, n. \tag{27}$$

This proves (21). Because of $s_2 = \sqrt{2}\partial_2 + s_1$ and $\Gamma^1_{ij} = 0$ it is

$$\omega_{12}(\partial_k) = \sqrt{2} g(\nabla_{\partial_k} s_1, \partial_2) = 0. \tag{28}$$

Furthermore, since $T = \partial_2$ is parallel, one obtains

$$\omega_{2I} - \omega_{1I} = g(\nabla(s_2 - s_1), s_I) = \sqrt{2} g(\nabla \partial_2, s_I) = 0. \tag{29}$$

Inserting (28) and (29) in (23) gives

$$\nabla^S_X \eta_\varepsilon = \frac{1}{\sqrt{2}} \sum_I \omega_{1I}(X) T \cdot s_I \cdot \eta_\varepsilon + \frac{1}{2} \sum_{I<J} \omega_{IJ}(X) s_I \cdot s_J \cdot \eta_\varepsilon. \tag{30}$$

Now, a direct calculation using Lemma 1 and (24)–(26) shows that

$$\begin{aligned}
\omega_{1I}(\partial_1) &= \frac{1}{\sqrt{2}}[(CPA\dot{R}^t)_I + \tfrac{1}{2}(CP\dot{A}R^t)_I], \\
\omega_{IJ}(\partial_1) &= (C\dot{P}AP^tC^t + \tfrac{1}{2}CP\dot{A}P^tC^t)_{IJ}, \\
\omega_{1I}(\partial_K) &= \frac{1}{\sqrt{2}}\left[\frac{1}{2}(CP\dot{A})_{IK} + \frac{1}{x_1 - y_1}(CP[E - AP^tA^{-1}(y_1)]W)_{IK}\right], \\
\omega_{IJ}(\partial_K) &= 0,
\end{aligned}$$

where $R = R(x)$ is the vector $R = \mathfrak{b}[A^{-1} - A^{-1}(y_1)P]$ and $\dot{R}$ denotes the derivative with respect to x_1. Hence with (30) this proves (20) and (22). The connection coefficients $\omega_{IJ}(\partial_1)$ and $\omega_{1I}(\partial_K)$ depend only on the first variable x_1. □

Proposition 11. *On the open submanifold $\{x \in M | x_1 \neq y_1\}$ the Dirac operator satisfies*

$$D(\eta_\varepsilon) = \sum_{I<J} f_{IJ} T \cdot s_I \cdot s_J \cdot \eta_\varepsilon + f T \cdot \eta_\varepsilon,$$

where f_{IJ} and f are functions depending only on x_1.

Proof. Since $\nabla^S_{\partial_2}\eta_\varepsilon = 0$ it is $\nabla^S_{s_1}\eta_\varepsilon = \nabla^S_{s_2}\eta_\varepsilon$. Hence

$$\begin{aligned} D(\eta_\varepsilon) &= \sum_i \varepsilon_i s_i \cdot \nabla^S_{s_i}\eta_\varepsilon = (s_2 - s_1)\cdot \nabla^S_{s_1}\eta_\varepsilon + \sum_I s_I \cdot \nabla^S_{s_I}\eta_\varepsilon \\ &= T\cdot(\nabla_{\partial_1}\eta_\varepsilon + (\mathrm{b}[A^{-1} - A^{-1}(y_1)P])_K \nabla^S_{\partial_K}\eta_\varepsilon) + \sum_{IK}(CP)_{IK} s_I \cdot \nabla^S_{\partial_K}\eta_\varepsilon. \end{aligned}$$

Using $T\cdot T = -g(T,T) = 0$ for the isotropic vector field T Proposition 10 gives

$$D(\eta_\varepsilon) = \frac{1}{2}\sum_{I<J}\omega_{IJ}(\partial_1)T\cdot s_I\cdot s_J\cdot\eta_\varepsilon - \frac{1}{\sqrt{2}}\sum_{IJK}(CP)_{IK}\omega_{1J}(\partial_K)T\cdot s_I\cdot s_J\cdot\eta_\varepsilon.$$

Since $P, \omega_{IJ}(\partial_1)$ and $\omega_{1J}(\partial_K)$ depend only on x_1, this proves the proposition. □

Proposition 12. *Let $h \in C^\infty(M)$ be a function depending only on x_1. Then*

$$D^2(h\eta_\varepsilon) = 0, \quad \varepsilon = 1,\dots,N.$$

Proof. Since all sections and functions are continuous it is enough to prove the assertion on the open submanifold $\{x \in M \mid x_1 \neq y_1\}$.

For the Dirac operator of a Lorentzian spin manifold the following commutation rules are valid

$$D(f\varphi) = fD\varphi + \operatorname{grad} f\cdot\varphi, \tag{31}$$

$$D(X\cdot\varphi) = -X\cdot D\varphi - 2\nabla^S_X\varphi + \sum_i \varepsilon_i s_i\cdot\nabla_{s_i}X\cdot\varphi, \tag{32}$$

where f is a function, X is a vector field and φ is a spinor field. If h is a function on the standard plane wave manifold depending only on x_1 then $\operatorname{grad} h = \dot{h}T$. Hence from (31) it follows

$$D(h\eta_\varepsilon) = \dot{h}T\cdot\eta_\varepsilon + hD\eta_\varepsilon.$$

Since $s_I\cdot s_J\cdot\eta_\varepsilon = A[\mathrm{IJ}]_{\varepsilon\delta}\eta_\delta$, where $A[\mathrm{IJ}]_{\varepsilon\delta}$ are constant functions given by the matrix representation of the linear map $e_I\cdot e_J\cdot$ on the spinor modul $\Delta_{n,1}$ with respect to the basis $(v_1,\dots,v_N)$, Proposition 11 results that $D(h\eta_\varepsilon)$ can be expressed in the form

$$D(h\eta_\varepsilon) = \tilde{h}_{\varepsilon\delta}T\cdot\eta_\delta,$$

where $\tilde{h}_{\varepsilon\delta}$ are functions depending only on x_1. Hence

$$D^2(h\eta_\varepsilon) = \dot{\tilde{h}}_{\varepsilon\delta}T\cdot T\cdot\eta_\delta + \tilde{h}_{\varepsilon\delta}D(T\cdot\eta_\delta).$$

Since T is parallel and $\nabla^S_T\eta_\varepsilon = 0$, from (32) it follows $D(T\cdot\varphi) = -T\cdot D\varphi$. Using $T\cdot T = -g(T,T) = 0$ this gives under consideration of Proposition 11

$$D^2(h\eta_\varepsilon) = -\tilde{h}_{\varepsilon\delta}T\cdot D\eta_\delta = 0.$$

□

Theorem 9. *The Hadamard coefficients $U_k \in \Gamma(M\times M, S^*\boxtimes S)$ of the square of the Dirac operator on a standard plane wave manifold vanish for all $k \geq 1$.*

Proof. From Proposition 3 it is known that the Hadamard coefficient U_0 satisfies

$$U_0(y,z) = \frac{1}{\tau(y,z)}\mathcal{P}(y,z),$$

where τ is the function described in Proposition 8 and $\mathcal{P}(y,z)$ is the parallel displacement from T_yM to T_zM along the geodesic joining y with z.

Since the basis $(s_1,\dots,s_n)$ is parallel along geodesics starting from y, the sections $\eta_\varepsilon = [\hat{s}, v_\varepsilon]$ in the spinor bundle are also parallel along these geodesics. Therefore, the parallel displacement $\mathcal{P}(y,z)$ can be expressed by

$$\mathcal{P}(y,z) = \sum_{\varepsilon} \eta_\varepsilon^*(y) \otimes \eta_\varepsilon(z).$$

According to Proposition 8 the function $\tau(y,\cdot)$ depends only on the first variable. Using Proposition 12 we obtain

$$D^2(U_0(y,\cdot)) = 0.$$

Then Proposition 3 shows that $U_1 = 0$. Hence all Hadamard coefficients U_k for $k \geq 1$ vanish. □

To be of Huygens type is a local property of an operator. Therefore, we obtain under consideration of Proposition 8 from the Hadamard criterion:

Theorem 10. *Let (M^n, g) be a plane wave spin manifold of even dimension $n \geq 4$. Then the square of the Dirac operator is of Huygens type.*

5. The Huygens property for twisted Dirac operators on four-dimensional Lorentzian spin manifolds

In this section we consider the Huygens property for twisted Dirac operators. Let (M^n, g) be a Lorentzian spin manifold with spinor bundle S. Furthermore, let P be a G-principal bundle over M, $\rho : G \longrightarrow GL(V)$ a complex representation of G and $E = P \times_\rho V$ the associated complex vector bundle. Each connection A of P induces a Dirac operator

$$D_A : \Gamma(S \otimes E) \longrightarrow \Gamma(S \otimes E)$$

with values in E defined by

$$D_A = D \otimes 1 + \sum_{i=1}^{n} \varepsilon_i s_i \cdot \otimes \nabla^A_{s_i}, \tag{33}$$

where D is the Dirac operator, ∇^A is the covariant derivative on E given by A, $\cdot$ is the Clifford multiplication and $(s_1,\dots,s_n)$ is a local orthonormal basis on (M,g). The Weitzenböck formula for D_A^2 is given by

$$D_A^2 = \Delta^\nabla + \tfrac{1}{4}R + Q_A, \tag{34}$$

where R is the scalar curvature of (M^n, g),

$$\nabla = \nabla^S \otimes 1 + 1 \otimes \nabla^A$$

and

$$Q_A = \sum_{i<j} \varepsilon_i \varepsilon_j s_i \cdot s_j \cdot \otimes F^A(s_i, s_j), \qquad F^A \text{ curvature of } \nabla^A$$

(see [LM89, p.164]). In even dimension n the spinor bundle S splits into the sum $S = S^+ \oplus S^-$ of positive and negative spinors. Let us denote by $D_A^\pm : \Gamma(S^\pm \otimes E) \longrightarrow \Gamma(S^\mp \otimes E)$ the restrictions of D_A to the corresponding subbundles. Then

$$D_A^2 = \begin{pmatrix} D_A^- D_A^+ & 0 \\ 0 & D_A^+ D_A^- \end{pmatrix} : \Gamma \begin{pmatrix} S^+ \otimes E \\ S^- \otimes E \end{pmatrix} \longrightarrow \Gamma(S^+ \otimes E \quad S^- \otimes E).$$

We first prove the following proposition.

Proposition 13. *Let (M^n, g) be a space and time oriented Lorentzian spin manifold of even dimension $n = 2m \geq 4$. Then the operator $D_A^- D_A^+ : \Gamma(S^+ \otimes E) \longrightarrow \Gamma(S^+ \otimes E)$ is of Huygens type if and only if the operator $D_A^+ D_A^- : \Gamma(S^- \otimes E) \longrightarrow \Gamma(S^- \otimes E)$ is of Huygens type.*

Proof. In [Bau94] it was proved that on the spinor bundle $S = S^+ \oplus S^-$ of an even-dimensional space and time oriented Lorentzian spin manifold there exists an antiunitary map $C : S^\pm \longrightarrow S^\mp$ such that

$$DC = -CD, \tag{35}$$

$$XC = -CX \quad \text{for all vector fields } X, \tag{36}$$

$$C^2 = (-1)^{\alpha(n)} Id, \quad \text{where } \alpha(n) = (m(m+1)/2) + 1. \tag{37}$$

If we extend C to $S \otimes E$ by $C(\phi \otimes e) = C\phi \otimes e$, from (35)–(37) it follows

$$D_A^+ D_A^- = (-1)^{\alpha(n)} C^+ D_A^- D_A^+ C^-. \tag{38}$$

Now, let Ω_0 be a causal domain and $x \in \Omega_0$. Let $G_+(x)$ and $G_-(x)$ be the forward and backward fundamental solutions of $D_A^+ D_A^-$ with respect to (Ω_0, x). We denote by $H_\pm(x) \in \mathcal{D}'_\pm(\Omega_0, (S^+ \otimes E)^*; (S^+ \otimes E)_x^*)$ the following distribution:

$$\langle H_\pm(x), \psi \rangle := (-1)^{\alpha(n)} (C^+)_x^* \langle G_\pm(x), (C^-)^* \psi \rangle, \qquad \psi \in \Gamma_0((S^+ \otimes E)^*),$$

where $(C^\pm)^*$ denotes the maps induced by $C^\pm$ on the dual bundles. Using (37) and (38) it is easy to check that $H_\pm(x)$ is the fundamental solution of $D_A^- D_A^+$ with respect to (Ω_0, x) and that the support of $H_\pm(x)$ coincides with that of $G_\pm(x)$. Hence the operator $D_A^+ D_A^-$ is Huygens if and only if $D_A^- D_A^+$ is Huygens. □

Now, we consider the Huygens property for the square D_A^2 of the twisted Dirac operator on a four-dimensional Lorentzian spin manifold. In case of $U(1)$-connections the following statement was proved by Illge ([Ill88]).

Theorem 11. *Let (M^4, g) be a space and time oriented analytic Lorentzian spin manifold of dimension 4, let f be a smooth function on M and $D_A : \Gamma(S \otimes E) \longrightarrow \Gamma(S \otimes E)$ be the Dirac operator coupled to a connection A of a complex vector bundle E. Let us denote by $P(A)$ one of the operators D_A^2, $D_A^+ D_A^-$ or $D_A^- D_A^+$. If $P(A) - f$ is of Huygens type, then the scalar curvature R of (M, g) is constant and equals $12f$.*

If R is constant and non-zero, then $P(A) - \frac{1}{12}R$ is of Huygens type if and only if the connection A is flat and (M^4, g) is of constant sectional curvature.

If $R = 0$, then $P(A)$ is of Huygens type if and only if A is flat and (M^4, g) is conformally flat or locally isometric to a plane wave manifold.

Proof. According to Proposition 13 it is enough to consider the case $P(A) = D_A^2$. Suppose that the operator $P(A) - f$ is of Huygens type. According to Theorem 4 this implies that its Cotton invariant C vanishes. From the Weitzenböck formula (34) of D_A^2 we obtain for the Cotton invariant $C = \frac{1}{12}R - f + Q_A$. Hence we have the condition

$$Q_A = \sum_{i<j} \varepsilon_i \varepsilon_j \, s_i \cdot s_j \cdot \otimes F^A(s_i, s_j) = (f - \tfrac{1}{12}R) Id_{S\otimes E}. \tag{39}$$

For the calculations we identify the complexified Clifford algebra $\mathrm{Cliff}_{4,1}^{\mathbb{C}}$ of the Minkowski space with the algebra of complex 4×4-matrices using the map Φ given by

$$\Phi(e_1) = \mathrm{i}E \otimes U, \quad \Phi(e_2) = E \otimes V, \quad \Phi(e_3) = U \otimes T, \quad \Phi(e_4) = V \otimes T,$$

where $(e_1, \ldots, e_4)$ denotes the canonical basis of the Minkowski space and

$$U = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

Let $u(\varepsilon) := \binom{1}{-\varepsilon \mathrm{i}}$, $\varepsilon = \pm 1$, and $u(\varepsilon_1, \varepsilon_2) := u(\varepsilon_1) \otimes u(\varepsilon_2)$. Let us denote by $\tilde{s}$ a lift of the local orthonormal frame $s = (s_1, \ldots, s_4)$ into the spinor structure and $\eta(\varepsilon_1, \varepsilon_2) := [\tilde{s}, u(\varepsilon_1, \varepsilon_2)]$ the corresponding local sections in the spinor bundle S. Then $(\eta(1, 1), \eta(-1, -1))$ is a local basis in S^+ and $(\eta(1, -1), \eta(-1, 1))$ is a local basis in S^-. If we use this basis, the homomorphism $Q_A^\pm$ is given by

$$Q_A^\pm \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} (\pm F_{12} + \mathrm{i}F_{34})\psi_1 + (\pm \mathrm{i}F_{13} \mp F_{14} - \mathrm{i}F_{23} + F_{24})\psi_2 \\ (\mp \mathrm{i}F_{13} + \mp F_{14} - \mathrm{i}F_{23} - F_{24})\psi_1 + (\mp F_{12} - \mathrm{i}F_{34})\psi_2 \end{pmatrix},$$

where $F_{ij} = F^A(s_i, s_j)$. This shows that Q_A^+ is a multiple of the identity if and only if $F^A = \mathrm{i} * F^A$ and Q_A^- is a multiple of the identity if and only if $F^A = -\mathrm{i} * F^A$. Here $*$ is the Hodge operator of the Lorentzian metric. Hence condition (39) implies that A is flat: $F^A = 0$. Since the Huygens property is a local one we can assume that M is simply connected. Then, because of the flatness of A, there exists a global trivialization of E by ∇^A- parallel sections. Using this trivialization the bundle $S \otimes E$ can be identified with the sum $(S \oplus \cdots \oplus S)$ of the spinor bundle such that the Dirac operator D_A acts on each factor S as the uncoupled Dirac operator D (see (33)). Hence $P(A) - f$ is Huygens if and only if A is flat and the uncoupled shifted Dirac operator $D^2 - f$ is Huygens. Then the assertion follows from the theorem of Wünsch (Theorem 1). □

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